Dynamics of a Spin-Stabilized Satellite Having Flexible Appendages

KAZUO TSUCHIYA* AND HARUO SAITO*
Mitsubishi Electric Co., Amagasaki, Hyogo, Japan

This paper deals with the dynamic features of a spin stabilized satellite having flexible appendages. The satellite is assumed to be constituted as a central rigid body and flexible appendages attached in the spin plane. The analysis is based on the method of averaging. The decay time constant for nutational body motion derived as the first-order solution of this method is the same as the result obtained by the energy sink method. The second-order decay time constants obtained by this method are in good agreement with numerically computed solutions over a range of parameter values for which corresponding results obtained by the energy sink method are not accurate. Analytical stability criteria are also obtained. Furthermore, the heavy damping characteristics of nutational body motion due to nonlinear internal resonance are investigated analytically by this method, and a closed form solution is obtained.

1. Introduction

NUMEROUS papers have been presented concerning the effect of internal energy dissipation on the attitude change of a satellite rotating freely in space. In most cases,[1–3] the energy sink method is used in the analysis. The basic assumption of this method is as follows: the relative motion causing an internal energy dissipation is considered to remove mechanical energy from a rigid body system without being coupled dynamically with the rigid body motion. It should be noted that, because of its analytical nature, this method gives a clear picture of the behavior of a freely spinning satellite with damping. However this method is heuristic and provides no quantitative measure of its range of application.

For example, this method is not appropriate in application to a spin-stabilized satellite having complex flexible appendages, for which energy dissipation results from the elastic deformations of the appendages induced by gyroscopic action. Moreover, when appendages are excited at near resonance conditions, this method is no longer applicable, because a large energy transfer takes place between the vibration of the appendages and nutational body motion, and the essential assumption of this method no longer holds. Therefore, new approaches suitable for exploring the basic characteristics of the dynamics of this class of satellite must be developed from an analytical viewpoint.

The attitude stability of this class of satellite has been investigated by several authors[4–9] from this viewpoint, and closed form stability criteria have been derived. The Liapunov's second method is used as the basic analytical tool. In this paper, the method of averaging[11] is employed to accomplish this goal.

In Sec. 3, we first deal with the linearized equations of motion. It is shown that the decay time constant for nutational body motion calculated as the first-order solution of the present method is rigorously the same as the result obtained by the energy sink method. Also, the second-order decay time constants show in good agreement with numerically computed solutions over a range of parameter values, for which the corresponding results obtained by the energy sink method are not accurate. The analytical stability criterion is deduced from the condition that the decay rate be negative. From the first-order solution, the maximum inertia axis criterion, as predicted by the energy sink method, is obtained. From the second-order solution, the stability condition involving the properties of flexible appendages such as the length and mass distribution, and the natural frequencies of the appendages, etc., is established.

In Sec. 4, heavy damping characteristics of nutational body motion due to a nonlinear internal resonance between the vibration of the appendages and nutational body motion are considered. Those phenomena have first been dealt with by Pringle.[10] His analysis, however, is based on the numerical integrations of equations of motion and a closed-form solution is not obtained. In this paper an analytical expression for the damping of nutational body motion is obtained using the method of averaging.

2. Fundamental Equations

Let us consider a satellite which is composed of a heavy central body and N light-weight appendages (Fig. 1). The reference axes (X_1, X_2, X_3) are assumed to be parallel to the principal axes of the undeformed total configuration (the X_3-axis coincides with the spin-axis), and O is the mass center of the undeformed total configuration. The mass center is assumed to remain fixed during small deformations of the appendages. For an appendage i, an axis system (ζ_i, η_i, ξ_i) is defined so that the appendage is coincident with the ζ_i-axis when it is undeflected and the ξ_i-axis is coincident with the spin-axis.

Let the angle of rotation from (X_1, X_2, X_3) to (ζ_i, η_i, ξ_i) be γ_i, and the angular velocity ω of the (X_1, X_2, X_3) axes has components $(\omega_1, \omega_2, \omega_3)$ in the (X_1, X_2, X_3) reference frame. If an arbitrary point on an appendage i, when deflected, is denoted by a vector γ_i, the vector γ_i is given by (ζ_i, U_i, V_i) in the (ζ_i, η_i, ξ_i) reference frame, where U_i and V_i are deflections in and perpendicular to the plane of the (ζ_i, η_i) axes, respectively. The total kinetic energy $\mathscr{T}$ is given by

$$\mathscr{T} = \tfrac{1}{2}(I_{11}'\omega_1^2 + I_{22}'\omega_2^2 + I_{33}'\omega_3^2 + 2I_{12}'\omega_1\omega_2) + \frac{1}{2}\sum_{i=1}^{N} \mu_i \int_0^{l_i} (\dot{\gamma}_i + \omega \times \gamma_i)^2 \, ds_i \qquad (1)$$

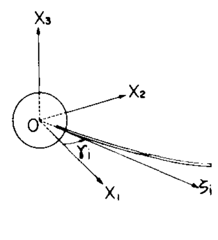

Fig. 1 Satellite configuration.

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* Research Scientist, Dynamics Division, Central Research Laboratory.

where $I_{11}', I_{22}', I_{33}'$ are the moments of inertia of the central body about the X_1, X_2, X_3 axes, respectively; I_{12}' is the product of inertia of the central body about the axes X_1, X_2; μ_i is the mass per unit length, which is assumed to be constant; ds_i is the arc length along the appendage i; and l_i is the total length of the appendage i. It is assumed that ω_1, ω_2, U_i and V_i are small.

Since

$$ds_i = [1+(\partial U_i/\partial \zeta_i)^2+(\partial V_i/\partial \zeta_i)^2]^{1/2}\, d\zeta_i$$

or, to the second order

$$ds_i = [1+\{(\partial U_i/\partial \zeta_i)^2+(\partial V_i/\partial \zeta_i)^2\}/2]\, d\zeta_i$$

it is seen that

$$\int_0^{l_i} \zeta_i^2\, ds_i = \int_0^{l_i} \zeta_i^2\, d\zeta_i - \frac{1}{2}\int_0^{l_i} (l_i^2-\zeta_i^2) \times \{(\partial U_i/\partial \zeta_i)^2+(\partial V_i/\partial \zeta_i)^2\}\, d\zeta_i$$

Then the total kinetic energy, Eq. (1), is expressed by the form

$$\begin{aligned}\mathscr{T} = {} & \tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+\\ & \frac{1}{2}\sum_{i=1}^{N} \mu_i[\langle \dot U_i^2\rangle+\langle \dot V_i^2\rangle+2\{\omega_1(S\gamma_i\langle \zeta_i \dot V_i\rangle-c\gamma_i\langle V_i \dot U_i\rangle+\\ & c\gamma_i\langle U_i \dot V_i\rangle)+\omega_2(S\gamma_i\langle U_i \dot V_i\rangle-S\gamma_i\langle \dot U_i V_i\rangle-c\gamma_i\langle \zeta_i \dot V_i\rangle)+\\ & \omega_3\langle \zeta_i \dot U_i\rangle\}+\{\omega_1^2 S2\gamma_i\langle \zeta_i U_i\rangle-\omega_2^2 S2\gamma_i\langle \zeta_i U_i\rangle+\\ & \omega_3^2\{\langle U_i^2\rangle-\tfrac{1}{2}\langle (l_i^2-\zeta_i^2)[(\partial U_i/\partial \zeta_i)^2+(\partial V_i/\partial \zeta_i)^2]\rangle\}-\\ & 2\omega_1\omega_2 c2\gamma_i\langle \zeta_i U_i\rangle-2\omega_2\omega_3(S\gamma_i\langle \zeta_i V_i\rangle+c\gamma_i\langle U_i V_i\rangle)-\\ & 2\omega_1\omega_3(c\gamma_i\langle \zeta_i V_i\rangle-S\gamma_i\langle U_i V_i\rangle)\}]\end{aligned} \tag{2}$$

where

$$I_1 = I_{11}'+\frac{1}{3}\sum_{i=1}^{N} \mu_i l_i^3 S^2\gamma_i, \quad I_2 = I_{22}'+\frac{1}{3}\sum_{i=1}^{N} \mu_i l_i^3 c^2\gamma_i$$

$$I_3 = I_{33}'+\frac{1}{3}\sum_{i=1}^{N} \mu_i l_i^3$$

are the moments of inertia of the undeformed configuration about the X_1, X_2, X_3 axes, respectively

$$\langle f_i\rangle = \int_0^{l_i} f_i\, d\zeta_i, \quad S\gamma_i = \sin\gamma_i, \ldots, c\gamma_i = \cos\gamma_i$$

and it has been assumed that

$$I_{12}'-\frac{1}{3}\sum_{i=1}^{N} \mu_i l_i^3 S\gamma_i c\gamma_i = 0$$

It is assumed that the external forces may be ignored in this analysis. The potential energy $\mathscr{U}$ consists entirely of the elastic strain energy of appendages, i.e.,

$$\mathscr{U} = \frac{1}{2}\sum_{i=1}^{N} B_i\{\langle(\partial^2 U_i/\partial \zeta_i^2)^2\rangle+\langle(\partial^2 V_i/\partial \zeta_i^2)^2\rangle\} \tag{3}$$

where B_i is the bending stiffness of the appendage i and is assumed to be constant.

The energy dissipation which results from elastic deformations of the appendages is represented by Rayleigh's dissipation function $\mathscr{F}$, which is given by

$$\mathscr{F} = \sum_{i=1}^{N} \mu_i \delta_i\{\langle \dot U_i^2\rangle+\langle \dot V_i^2\rangle\} \tag{4}$$

where δ_i is the damping ratio for the appendage i. U_i and V_i are expanded in terms of the normal modes associated with a cantilever, i.e.,

$$U_i = l_i \sum_{n=1}^{\infty} P_{in}(t) E_n(\hat\zeta) \tag{5a}$$

$$V_i = l_i \sum_{n=1}^{\infty} T_{in}(t) E_n(\hat\zeta) \tag{5b}$$

where the normal modes $E_n(\hat\zeta)$ satisfy the differential equation

$$(d^4/d\hat\zeta^4)E_n(\hat\zeta)-\lambda_n^4 E_n(\hat\zeta) = 0 \tag{6}$$

and the boundary conditions

$$\left.\begin{aligned} & E_n = 0, \quad (d/d\hat\zeta)E_n(\hat\zeta) = 0 \quad \text{at} \quad \hat\zeta = 0\\ & (d^2/d\hat\zeta^2)E_n(\hat\zeta) = 0, \quad (d^3/d\hat\zeta^3)E_n(\hat\zeta) = 0 \quad \text{at} \quad \hat\zeta = 1\end{aligned}\right\} \tag{7}$$

In addition, they are normalized such that

$$\int_0^1 E_n \cdot E_m\, d\hat\zeta = \delta_{nm}$$

where δ_{nm} is the Kronecker delta, λ_n is the natural frequency of the normal mode n and $\hat\zeta = \zeta_i/l_i$. As shown later, the appendages are excited near or below the first resonance frequency, so we may truncate the series expression, (5), at the first mode, i.e., $n = 1$.

Substituting Eq. (5) into Eqs. (2–4) and neglecting the suffix 1, we obtain

$$\begin{aligned}\mathscr{T} = {} & \tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)+\frac{1}{2}\sum_{i=1}^{N} \mu_i l_i^3[(\dot T_i^2+\dot P_i^2)+\\ & 2\{\omega_1(S\gamma_i \varepsilon \dot T_i+c\gamma_i \dot T_i P_i-c\gamma_i T_i \dot P_i)+\omega_2(-c\gamma_i \varepsilon \dot T_i+\\ & S\gamma_i P_i \dot T_i-S\gamma_i T_i \dot P_i)+\omega_3 \varepsilon \dot P_i\}+\{\omega_1^2 S2\gamma_i \varepsilon P_i-\\ & \omega_2^2 S2\gamma_i \varepsilon P_i+\omega_3^2[P_i^2-\beta(T_i^2+P_i^2)]-\\ & 2\omega_1\omega_2 c2\gamma_i \varepsilon P_i-2\omega_2\omega_3(S\gamma_i \varepsilon T_i+c\gamma_i P_i T_i)-\\ & 2\omega_1\omega_3(c\gamma_i \varepsilon T_i-S\gamma_i P_i T_i)\}]\end{aligned} \tag{8a}$$

$$\mathscr{U} = \frac{1}{2}\sum_{i=1}^{N} (B_i \lambda^4/l_i)(T_i^2+P_i^2) \tag{8b}$$

$$\mathscr{F} = \frac{1}{2}\sum_{i=1}^{N} \mu_i l_i^3 \delta_i(\dot T_i^2+\dot P_i^2) \tag{8c}$$

where

$$\varepsilon = \int_0^1 \hat\zeta E(\hat\zeta)\, d\hat\zeta = -0.5688$$

$$\beta = \frac{1}{2}\int_0^1 (1-\hat\zeta^2)[(d/d\hat\zeta)E(\hat\zeta)]^2\, d\hat\zeta = 1.193$$

The Lagrangian equations of motion for T_i and P_i have the forms

$$(d/dt)(\partial \mathscr{L}/\partial \dot T_i)-(\partial \mathscr{L}/\partial T_i) = -(\partial \mathscr{F}/\partial \dot T_i) \tag{9a}$$

$$(d/dt)(\partial \mathscr{L}/\partial \dot P_i)-(\partial \mathscr{L}/\partial P_i) = -(\partial \mathscr{F}/\partial \dot P_i) \tag{9b}$$

where the Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L} = \mathscr{T} - \mathscr{U} \tag{10}$$

Since the coordinates ω_1, ω_2, ω_3 are so-called quasi-coordinates, the corresponding equations of motion are

$$\left.\begin{aligned} & (d/dt)(d\mathscr{T}/\partial\omega_1)+\omega_2(\partial\mathscr{T}/\partial\omega_3)-\omega_3(\partial\mathscr{T}/\partial\omega_2) = N_1\\ & (d/dt)(\partial\mathscr{T}/\partial\omega_2)+\omega_3(\partial\mathscr{T}/\partial\omega_1)-\omega_1(\partial\mathscr{T}/\partial\omega_3) = N_2\\ & (d/dt)(\partial\mathscr{T}/\partial\omega_3)+\omega_1(\partial\mathscr{T}/\partial\omega_2)-\omega_2(\partial\mathscr{T}/\partial\omega_1) = N_3\end{aligned}\right\} \tag{11}$$

where N_1, N_2, N_3 are the torque components about the X_1, X_2, X_3 axes, respectively.

When the external torques are neglected, $N_1 = N_2 = N_3 = 0$, and the equations of motion may be written as follows:

$$\left.\begin{aligned} & I_1\dot\omega_1+(I_3-I_2)\omega_2\omega_3+\sum_{i=1}^{N} \mu_i l_i^3\varepsilon\{S\gamma_i(\ddot T_i+\omega_3^2 T_i)+\\ & \quad S2\gamma_i(\dot P_i\omega_1+P_i\dot\omega_1)-c2\gamma_i(\dot P_i\omega_2+P_i\dot\omega_2)+\omega_2\dot P_i+\\ & \quad c2\gamma_i\omega_1\omega_3 P_i+S2\gamma_i\omega_2\omega_3 P_i-c\gamma_i\dot\omega_3 T_i\} = 0\\ & I_2\dot\omega_2-(I_3-I_1)\omega_1\omega_3+\sum_{i=1}^{N} \mu_i l_i^3\varepsilon\{-c\gamma_i(\ddot T_i+\omega_3^2 T_i)-\\ & \quad c2\gamma_i(\dot P_i\omega_1+P_i\dot\omega_1)-S2\gamma_i(\dot P_i\omega_2+P_i\dot\omega_2)-\omega_1\dot P_i+\\ & \quad S2\gamma_i\omega_1\omega_3 P_i-c2\gamma_i\omega_2\omega_3 P_i-S\gamma_i\dot\omega_3 T_i\} = 0\\ & I_3\dot\omega_3+(I_2-I_1)\omega_1\omega_2+\sum_{i=1}^{N} \mu_i l_i^3\varepsilon\{\ddot P_i-c\gamma_i(\dot T_i\omega_1+T_i\dot\omega_1)-\\ & \quad S\gamma_i(\dot T_i\omega_2+T_i\dot\omega_2)-(c\gamma_i\dot T_i\omega_1+S\gamma_i\omega_1\omega_3 T_i)-\\ & \quad (S\gamma_i\omega_2\dot T_i-c\gamma_i\omega_2\omega_3 T_i) = 0\\ & \ddot T_i+2\delta_i\dot T_i+k_{T_i}^2 T_i+S\gamma_i\varepsilon(\dot\omega_1+\omega_2\omega_3)-c\gamma_i\varepsilon(\dot\omega_2-\omega_1\omega_3)+\\ & \quad c\gamma_i(2\dot P_i\omega_1+P_i\dot\omega_1)+s\gamma_i(2\dot P_i\omega_2+P_i\dot\omega_2) = 0\\ & \qquad (i = 1,\ldots,N)\\ & \ddot P_i+2\delta_i\dot P_i+k_{P_i}^2 P_i+\varepsilon\omega_3-(1/2)\varepsilon(S2\gamma_i\omega_1^2-2c2\gamma_i\omega_1\omega_2-\\ & \quad S2\gamma_i\omega_2^2)-c\gamma_i(2\dot T_i\omega_1+T_i\dot\omega_1)-S\gamma_i(2\dot T_i\omega_2+T_i\dot\omega_2)-\\ & \quad S\gamma_i\omega_1\omega_3 T_i+c\gamma_i\omega_2\omega_3 T_i = 0 \qquad (i = 1,\ldots,N)\end{aligned}\right\} \tag{12}$$

where

$$k_{T_i}{}^2 = \{\lambda^4 B_i/(\mu_i l_i^4)\} + \beta\omega_3{}^2, \quad k_{P_i}{}^2 = \{\lambda^4 B_i/(\mu_i l_i^4)\} + (\beta - 1)\omega_3{}^2$$

The motion may be classified into two modes; the first mode is the deflections of the appendages induced by the nutational body motion and the second mode is the nutational body motion induced by the vibrations of the appendages. Since the second mode has no practical importance, only the first mode of motion is investigated in this paper.

Without loss of generality the initial conditions may be taken as

$$\left.\begin{aligned} \omega_1 = \omega_{10}, \quad \omega_2 = 0, \quad T_i = 0, \quad \dot{T}_i = 0 \\ P_i = 0, \quad \dot{P}_i = 0, \quad \omega_3 = \omega_o \end{aligned}\right\} \text{at } t = 0 \tag{13}$$

3. Analysis of Linearized Equations

When the analysis is confined to small elastic motion of the appendages and nutational body motion, Eqs. (12) can be linearized as follows:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_o\omega_2 = -\sum_{i=1}^{N} \mu_i l_i^3 \varepsilon S\gamma_i(\ddot{T}_i + \omega_o{}^2 T_i) \tag{14a}$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_o\omega_1 = \sum_{i=1}^{N} \mu_i l_i^3 \varepsilon c\gamma_i(\ddot{T}_i + \omega_o{}^2 T_i) \tag{14b}$$

$$\ddot{T}_i + 2\delta_i\dot{T}_i + k_{T_i}{}^2 T_i = c\gamma_i\varepsilon(\dot{\omega}_2 - \omega_o\omega_1) - \varepsilon S\gamma_i(\dot{\omega}_1 + \omega_o\omega_2) \quad (i = 1, \ldots, N) \tag{14c}$$

$$I_3\dot{S} + \sum_{i=1}^{N} \mu_i l_i^3 \varepsilon \ddot{P}_i = 0 \tag{15a}$$

$$\ddot{P}_i + 2\delta_i\dot{P}_i + k_{P_i}{}^2 P_i = -\varepsilon\dot{S} \qquad (i = 1, \ldots, N) \tag{15b}$$

where it is assumed that $\omega_3 = \omega_o + S$; ω_o is a constant and S is a small variation.

Linearized equations of motion fall into two uncoupled sets[5]: the first set, which describes nutational body motion and appendage deformation perpendicular to the spin plane, and the second set, which describes the change in spin rate and the deflection of the appendages in the spin plane. The second set is independent of nutational body motion and so is not of direct interest in the present analysis. For this reason, in the remainder of this paper, the first set is further studied.

Solving Eq. (14c) for T_i under the conditions (13), we obtain

$$T_i = \varepsilon/(2i\hat{k}_{T_i})\left[\int_0^t \{c\gamma_i(\dot{\omega}_2 - \omega_o\omega_1) - S\gamma_i(\dot{\omega}_1 + \omega_o\omega_2)\} \times \exp(i\hat{k}_{T_i} - \delta_i)(t - t')\,dt'\right] + \text{complex conjugate part} \equiv \varepsilon\mathscr{L}_{ir}(\omega_1, \omega_2, t) \tag{16}$$

where

$$\hat{k}_{T_i} = (k_{T_i}{}^2 - \delta_i{}^2)^{1/2}$$

Substituting these expressions for T_i into Eqs. (14a) and (14b), we obtain the following integro-differential equations:

$$\left.\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_o\omega_2 &= -\sum_{i=1}^{N} \mu_i l_i^3 \varepsilon^2 S\gamma_i\{(d^2/dt^2) + \omega_o{}^2\} \times \mathscr{L}_{ir}(\omega_1, \omega_2, t) \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_o\omega_1 &= \sum_{i=1}^{N} \mu_i l_i^3 \varepsilon^2 c\gamma_i\{(d^2/dt^2) + \omega_o{}^2\} \times \mathscr{L}_{ir}(\omega_1, \omega_2, t) \end{aligned}\right\} \tag{17}$$

Since ε^2 is small, it may conveniently be used as a small parameter in obtaining a solution of Eqs. (17) by the method of averaging.[11] Let us introduce new variable a by the following transformation:

$$\left.\begin{aligned} \omega_1 &= \hat{\omega}_1(a\,e^{i\hat{\alpha}t} + a^*e^{-i\hat{\alpha}t}) \\ \omega_2 &= -i\hat{\omega}_2(a\,e^{i\hat{\alpha}t} - a^*e^{-i\hat{\alpha}t}) \end{aligned}\right\} \tag{18}$$

where

$$\hat{\omega}_1 = (M/I_1)\{(I_3/I_2) - 1\}^{1/2}, \quad \hat{\omega}_2 = (M/I_2)\{(I_3/I_1) - 1\}^{1/2}$$

$$\hat{\alpha} = \omega_o[\{(I_3/I_1) - 1\}\{(I_3/I_2) - 1\}]^{1/2}$$

M is a small constant, and a^* is a complex conjugate of a. Note the resulting necessary condition for stability

$$\{(I_3/I_1) - 1\}\{(I_3/I_2) - 1\} > 0 \tag{19}$$

The following forms of expansion are assumed:

$$\left.\begin{aligned} a &= \hat{a} + \sum_{P=1}^{\infty} \varepsilon^{2P} F_{\hat{a}}{}^{(P)}(\hat{a}, \hat{a}^*) F_t{}^{(P)}(t) \\ \dot{\hat{a}} &= \sum_{q=1}^{\infty} \varepsilon^{2q} G_{\hat{a}}{}^{(q)}(\hat{a}, \hat{a}^*) \end{aligned}\right\} \tag{20}$$

Substituting Eqs. (20) into Eq. (16) and carrying out the integration by parts, we can express $\mathscr{L}_{ir}$ as power series in ε^2

$$\begin{aligned} \mathscr{L}_{ir} = {} & [1/(2i\hat{k}_{T_i})]\{[f_{1i}\hat{a}\,e^{i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} + i\hat{\alpha}) + \\ & f_{1i}{}^*\hat{a}^* e^{-i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha})] + \varepsilon^2[[f_{2i}G_{\hat{a}}{}^{(1)} e^{i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} + i\hat{\alpha}) + \\ & f_{2i}{}^* G_{\hat{a}}{}^{(1)*} e^{-i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha})] - [f_{1i}G_{\hat{a}}{}^{(1)} e^{i\hat{\alpha}t}/ \\ & (\delta_i - i\hat{k}_{T_i} + i\hat{\alpha})^2 + f_{1i}{}^* G_{\hat{a}}{}^{(1)*} e^{-i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha})^2] + \\ & e^{(i\hat{k}_{T_i} - \delta_i)t}(f_{1i}F_{\hat{a}}{}^{(1)} \textstyle\int^t F_t{}^{(1)} e^{(\delta_i - i\hat{k}_{T_i} + i\hat{\alpha})t'}\,dt' + \\ & f_{1i}{}^* F_{\hat{a}}{}^{(1)*} \textstyle\int^t F_t{}^{(1)*} e^{(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha})t'}\,dt') + e^{(i\hat{k}_{T_i} - \delta_i)t} \times \\ & (f_{2i}F_{\hat{a}}{}^{(1)} \textstyle\int^t \dot{F}_t{}^{(1)} e^{(\delta_i - i\hat{k}_{T_i} + i\hat{\alpha})t'}\,dt' + \\ & f_{2i}{}^* F_{\hat{a}}{}^{(1)*} \textstyle\int^t \dot{F}_t{}^{(1)*} e^{(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha})t'}\,dt')]\} + \\ & \text{complex conjugate part} + O(\varepsilon^4) \end{aligned} \tag{21}$$

where

$$f_{1i} = c\gamma_i(\hat{\omega}_2\hat{\alpha} - \omega_o\hat{\omega}_1) - iS\gamma_i(\hat{\omega}_1\hat{\alpha} - \omega_o\hat{\omega}_2),$$

$$f_{2i} = -(S\gamma_i\hat{\omega}_1 + ic\gamma_i\hat{\omega}_2)$$

Substituting Eqs. (20) (21) into Eqs. (17) and rearranging the terms of the same order of magnitude, we obtain a series of equations as follows:

$$\begin{aligned} G_{\hat{a}}{}^{(1)} + F_{\hat{a}}{}^{(1)}\dot{F}_t{}^{(1)} = {} & \sum_{i=1}^{N} \mu_i l_i^3(\omega_o{}^2 - \hat{\alpha}^2)[iS\gamma_i/(I_1\hat{\omega}_1) + \\ & c\gamma_i/(I_2\hat{\omega}_2)]/(4\hat{k}_{T_i})\{f_{1i}\hat{a}/(\delta_i - i\hat{k}_{T_i} + i\hat{\alpha}) - \\ & f_{1i}{}^*\hat{a}^* e^{2i\hat{\alpha}t}/(\delta_i + i\hat{k}_{T_i} - i\hat{\alpha}) + f_{1i}{}^*\hat{a}^* e^{-2i\hat{\alpha}t}/(\delta_i - i\hat{k}_{T_i} - i\hat{\alpha}) - \\ & f_{1i}\hat{a}/(\delta_i + i\hat{k}_{T_i} + i\hat{\alpha})\} \end{aligned} \tag{22}$$

...

The crucial point of the method of averaging consists of the effective use of an averaging operation. In this paper the following averaging operation may suitably be chosen

$$\bar{\Phi}(\hat{a}, \hat{a}^*) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(\hat{a}, \hat{a}^*, t)\,dt \tag{23}$$

With this choice, we have in nth order

$$G_{\hat{a}}{}^{(n)} = \overline{\text{RHS}} \tag{24a}$$

[where RHS denotes the right-hand side of Eqs. (22)] and

$$F_{\hat{a}}{}^{(n)}\dot{F}_t{}^{(n)} = \text{RHS} - \overline{\text{RHS}} \tag{24b}$$

with the additional condition

$$\overline{F_a{}^{(n)}F_t{}^{(n)}} = 0 \tag{24c}$$

Application of this procedure to Eq. (22) leads to the following first-order equations

$$\left.\begin{aligned} a &= \hat{a} \\ \dot{\hat{a}} &= \varepsilon^2 G_{\hat{a}}{}^{(1)} \\ G_{\hat{a}}{}^{(1)} &= \sum_{i=1}^{N} \Pi_i(h_{1i} + ih_{2i})\hat{a} \end{aligned}\right\} \tag{25}$$

where

$$\Pi_i = \mu_i l_i^3 f_{1i} f_{1i}{}^*(\omega_o{}^2 - \hat{\alpha}^2)/\{4\hat{k}_{T_i}\hat{\omega}_1\hat{\omega}_2\omega_o(I_3 - I_1 - I_2)\}$$

$$h_{1i} = \{1/[\delta_i{}^2 + (\hat{k}_{T_i} - \hat{\alpha})^2] - 1/[\delta_i{}^2 + (\hat{k}_{T_i} + \hat{\alpha})^2]\}\,\delta_i$$

$$h_{2i} = \{(\hat{k}_{T_i} - \hat{\alpha})/[\delta_i{}^2 + (\hat{k}_{T_i} - \hat{\alpha})^2] + (\hat{k}_{T_i} + \hat{\alpha})/[\delta_i{}^2 + (\hat{k}_{T_i} + \hat{\alpha})^2]\}$$

and

$$F_{\hat{a}}{}^{(1)}F_t{}^{(1)} = (1/2i\hat{\alpha})\sum_{i=1}^{N} \Pi_i(f_{1i}{}^*/f_{1i})(h_{1i} - ih_{2i})e^{-2i\hat{\alpha}t}\hat{a}^* \tag{26}$$

Then, we find for the first-order damping ratio $\delta_1{}^*$ for nutational body motion

$$\delta_1{}^* = -\varepsilon^2 \sum_{i=1}^{N} \Pi_i h_{1i} \tag{27}$$

It should be noted that the first-order solution, Eq. (27), is the same as that obtained by the energy sink method. Stability requires that the decay rate be negative. This in turn requires that

$$\hat{\omega}_1 \hat{\omega}_2 > 0 \tag{28}$$

The combination of Eqs. (19) and (28), as predicted by the energy sink method, requires that the spin axis be the axis of maximum moment of inertia, i.e.,

$$I_3 > I_1, \qquad I_3 > I_2 \tag{29}$$

In the second approximation, the equation for the approximate amplitude $\hat{a}$ of the nutational body motion is obtained as follows:

$$\dot{\hat{a}} = \left[\varepsilon^2 \sum_{i=1}^{N} \Pi_i (h_{1i} + ih_{2i}) + \varepsilon^4 \sum_{i,j=1}^{N} \Pi_i \Pi_j (h_{1j} + ih_{2j}) \times \{\Gamma_{ij}(h_{1i}+ih_{2i}) - (g_{1i}+ig_{2i})\}\right] \hat{a} \tag{30}$$

where

$$g_{1j} = \{\delta_j{}^2 - (\hat{k}_{T_j} - \hat{\alpha})^2\}/\{\delta_j{}^2 + (\hat{k}_{T_j} - \hat{\alpha})^2\}^2 - \{\delta_j{}^2 - (\hat{k}_{T_j} + \hat{\alpha})^2\}/\{\delta_j{}^2 + (\hat{k}_{T_j} + \hat{\alpha})^2\}^2$$

$$g_{2j} = 2\delta_j(\hat{k}_{T_j} - \hat{\alpha})/\{\delta_j{}^2 + (\hat{k}_{T_j} - \hat{\alpha})^2\}^2 + 2\delta_j(\hat{k}_{T_j} + \hat{\alpha})/\{\delta_j{}^2 + (\hat{k}_{T_j} + \hat{\alpha})^2\}^2$$

$$\Gamma_{ij} = 2\hat{\alpha} i/(\omega_o{}^2 - \hat{\alpha}^2) + if_{1i}{}^* f_{1j}/(2\hat{\alpha} f_{1i} f_{1j}{}^*) + (f_{2i} f_{1j}{}^* - f_{2i}{}^* f_{1j})/(f_{1i} f_{1j}{}^*)$$

Then, the second-order damping ratio $\delta_2{}^*$ for nutational body motion is

$$\delta_2{}^* = -\varepsilon^2 \sum_{i=1}^{N} \Pi_i h_{1i} + \varepsilon^4 \sum_{i,j=1}^{N} \Pi_i \Pi_j \{(\Gamma_{ij})_i (h_{1j} h_{2i} + h_{2j} h_{1i}) - (\Gamma_{ij})_r (h_{1j} h_{1i} - h_{2j} h_{2i}) + (h_{1j} g_{1i} - h_{2j} g_{2i})\} \tag{31}$$

where

$$\Gamma_{ij} = (\Gamma_{ij})_r + i(\Gamma_{ij})_i$$

The analytical stability criterion deduced from Eq. (31) gives the following result:

$$\delta_2{}^* \geqq 0 \text{ implies stability}, \quad \delta_2{}^* < 0 \text{ implies instability} \tag{32}$$

As an example, a symmetrical satellite composed of a central body and four equal length appendages of identical material is investigated. The results are shown in Fig. 2. The decay time constant for nutational body motion calculated on the basis of the energy sink method, which can be rigorously derived as the first-order solution of the present method is considerably different from the results obtained by numerical computation. However, it is found that the decay time constants derived as the second-order solutions of the present method are rather close to the results obtained by numerical computation. The criterion Eq. (32), has been checked for a number of critical cases against results obtained on the basis of digital computer eigenvalue analyses and more rigorous methods.[5-9] Results presented in Fig. 3 show that qualitative agreement is found.

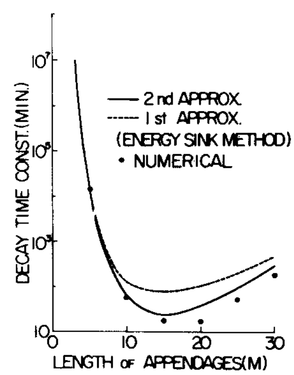

Fig. 2 Decay time constants.

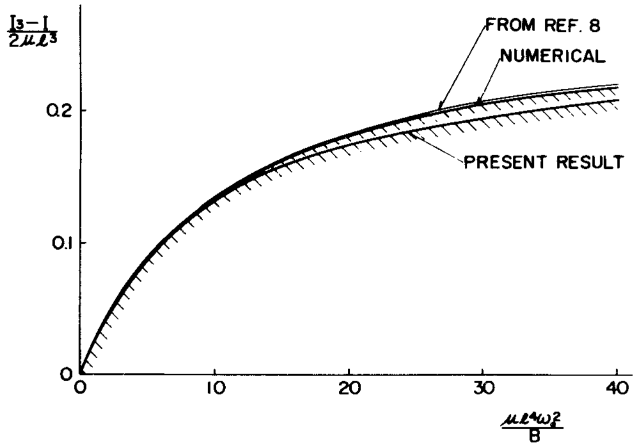

Fig. 3 Stability regions (hatched regions are unstable).

4. Analysis of Nonlinear Equations

In this section we consider the damping characteristics of nutational body motion due to a resonance between the vibration of the appendages and the nutational body motion. In the case of the linearized equations, Eqs. (14), this phenomenon does not occur because the natural frequencies of the lateral vibration of the appendages are always large in comparison with the nutational frequency $\hat{\alpha}$. However, in the case of the higher order approximation corresponding to Eqs. (12), a certain resonance phenomenon between the vibration of the appendages in the spin plane and nutational body motion may occur. For the sake of simplicity, in this section a symmetrical satellite composed of a central rigid body and four equal length flexible appendages of identical material is investigated. The deflections of appendages perpendicular to the spin plane are neglected. Then Eqs. (12) become

$$I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2 + \varepsilon\mu l^3 \sum_{i=1}^{4} \{-c2\gamma_i(\dot{P}_i\omega_2 + P_i\dot{\omega}_2) + \omega_2\dot{P}_i + c2\gamma_i\omega_3\omega_1 P_i\} = 0 \tag{33a}$$

$$I\dot{\omega}_2 - (I_3 - I)\omega_3\omega_1 + \varepsilon\mu l^3 \sum_{i=1}^{4} \{-c2\gamma_i(\dot{P}_i\omega_1 + P_i\dot{\omega}_1) - \omega_1\dot{P}_i - c2\gamma_i\omega_3\omega_2 P_i\} = 0 \tag{33b}$$

$$I_3\dot{S} + \varepsilon\mu l^3 \sum_{i=1}^{4} \ddot{P}_i = 0 \tag{33c}$$

$$\ddot{P}_i + 2\delta\dot{P}_i + k_P{}^2 P_i + \varepsilon\dot{S} + \varepsilon c2\gamma_i \omega_1\omega_2 = 0 \qquad (i = 1, \ldots, 4) \tag{33d}$$

where

$$I_1 = I_2 = I, \quad \gamma_i = (i-1)\pi/2 \quad (i = 1, \ldots, 4),$$
$$k_P{}^2 = \lambda^4 B/(\mu l^4) + (\beta - 1)\omega_o{}^2$$

The initial conditions are

$$\omega_1 = \omega_{10}, \quad \omega_2 = 0, \quad S = 0, \quad P = 0, \quad \dot{P} = 0 \text{ at } t = 0 \tag{34}$$

From Eqs. (33c) and (33d), and the initial conditions, Eqs. (34), we obtain

$$\sum_{i=1}^{4} P_i = 0, \quad S = 0 \tag{35}$$

and a solution may be obtained in the form

$$P = \sum_{i=1}^{4} c2\gamma_i P_i \tag{36}$$

Then, Eqs. (33) become as follows:

$$I\dot{\omega}_1 + (I_3 - I)\omega_o\omega_2 = \varepsilon\mu l^3[(\dot{P}\omega_2 + P\dot{\omega}_2) - \omega_o\omega_1 P] \tag{37a}$$

$$I\dot{\omega}_2 - (I_3 - I)\omega_o\omega_1 = \varepsilon\mu l^3[(\dot{P}\omega_1 + P\dot{\omega}_1) + \omega_o\omega_2 P] \tag{37b}$$

$$\ddot{P} + 2\delta\dot{P} + k_P{}^2 P = -4\varepsilon\omega_1\omega_2 \tag{37c}$$

First these equations are treated on the basis of the above-mentioned method. We seek a solution in the form

$$\omega_1 = \hat{\omega}(a\,e^{i\hat{\alpha}} + a^*\,e^{-i\hat{\alpha}})$$
$$\omega_2 = -i\hat{\omega}(a\,e^{i\hat{\alpha}} - a^* e^{-i\hat{\alpha}})$$

where $\hat{\alpha} = \omega_o(I_3/I - 1)$ and $\hat{\omega}$ is a small constant. The equation for the approximate amplitude $\hat{a}$ of nutational body motion is obtained after a simple, though lengthy, calculation. Setting the approximate amplitude at

$$\hat{a} = r\,e^{i\Theta} \tag{38}$$

we have

$$\dot{r} = -2\varepsilon^2\mu l^3\hat{\omega}^2(\omega_o+\hat{\alpha})\delta r^3\{1/[\delta^2+(\hat{k}_P-2\hat{\alpha})^2] - 1/[\delta^2+(\hat{k}_P+2\hat{\alpha})^2]\}/(I\hat{k}_P) \tag{39a}$$

$$\dot{\Theta} = -2\varepsilon^2\mu l^3\hat{\omega}^2(\omega_o+\hat{\alpha})r^2\{(\hat{k}_P-2\hat{\alpha})/[\delta^2+(\hat{k}_P-2\hat{\alpha})^2] + (\hat{k}_P+2\hat{\alpha})/[\delta^2+(\hat{k}_P+2\hat{\alpha})^2]\}/(I\hat{k}_P) \tag{39b}$$

where

$$\hat{k}_P = (k_P{}^2 - \delta^2)^{1/2}$$

The solution for r is

$$r/r_{oi} = 1/[\![1+4\varepsilon^2\mu l^3\hat{\omega}^2(\omega_o+\hat{\alpha})\delta r_{oi}{}^2 t\{1/[\delta^2+(\hat{k}_P-2\hat{\alpha})^2] - 1/[\delta^2+(\hat{k}_P+2\hat{\alpha})^2]\}/(I\hat{k}_P)]\!]^{1/2} \tag{40}$$

where r_{oi} is the initial value of r.

Equation (40) describes the damping of nutational body motion due to nonlinear vibrations of the appendages. This solution is expected to be approximately valid whenever the right-hand sides of Eqs. (39) are "small." Inaccuracy of the result obtained by the above method (also, by the energy sink method) may arise when appendages are excited at "near resonance" conditions, because in this case the right-hand sides of Eqs. (39) become very large.

When the near resonance conditions are satisfied, i.e.,

$$\delta/k_P, \quad (k_P - 2\hat{\alpha})/k_P \lesssim 0(\varepsilon\hat{\omega}/k_P)^2 \tag{41}$$

a large nonlinear energy transfer takes place between the vibration of the appendages and nutational body motion, and the dissipation of energy causes a large damping of nutational body motion. These phenomena can be investigated by the following analysis. Let

$$\omega_1 = \hat{\omega}(a\,e^{ik_Pt/2} + a^*\,e^{-ik_Pt/2}) \tag{42a}$$
$$\omega_2 = -i\hat{\omega}(a\,e^{ik_Pt/2} - a^*\,e^{-ik_Pt/2}) \tag{42b}$$
$$P = c\,e^{ik_Pt} + c^*\,e^{-ik_Pt} \tag{42c}$$

and

$$\dot{c}\,e^{ik_Pt} + \dot{c}^*\,e^{-ik_Pt} = 0 \tag{43}$$

then Eqs. (37) become

$$\dot{a} = -i\Delta(a - a^*\,e^{-ik_Pt}) + (\mu l^3\varepsilon/I)\{-(k_P/2+\omega_o)a^*c + (3k_P/2-\omega_o)\times a^*c^*\,e^{-2ik_Pt} + i(\dot{a}^*c + a^*\dot{c}) + i(\dot{a}^*c^* + a^*\dot{c}^*)\,e^{-2ik_Pt}\} \tag{44a}$$

$$\dot{c} = (1/k_P)\{2\hat{\omega}^2\varepsilon(a^2 - a^{*2}\,e^{-2ik_Pt}) - \delta k_P(c - c^*\,e^{-2ik_Pt})\} \tag{44b}$$

where

$$\Delta = k_P/2 - \hat{\alpha}$$

The following forms of expansion are assumed:

$$\left.\begin{aligned}
a &= \hat{a} + \sum_{m=1}^{\infty} \varepsilon^m F_{\hat{a}}{}^{(m)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*, t)\\
c &= \hat{c} + \sum_{m=1}^{\infty} \varepsilon^m F_{\hat{c}}{}^{(m)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*, t)\\
\dot{\hat{a}} &= \sum_{n=1}^{\infty} \varepsilon^n G_{\hat{a}}{}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*)\\
\dot{\hat{c}} &= \sum_{n=1}^{\infty} \varepsilon^n G_{\hat{c}}{}^{(n)}(\hat{a}, \hat{a}^*, \hat{c}, \hat{c}^*)
\end{aligned}\right\} \tag{45}$$

Substituting Eqs. (45) into Eqs. (44), taking into account the relations

$$\delta/k_P, \quad \Delta/k_P \lesssim O[(\varepsilon\hat{\omega}/k_P)^2] \tag{46}$$

and arranging the terms of the same order of magnitude, we obtain a series of equations for $G_{\hat{a}}{}^{(n)}$, $G_{\hat{c}}{}^{(n)}$, $F_{\hat{a}}{}^{(n)}$ and $F_{\hat{c}}{}^{(n)}$. With the averaging operation given by Eq. (23), we obtain the following second-order equations for $\hat{a}$ and $\hat{c}$:

$$\dot{\hat{a}} = -\varepsilon\alpha_o\hat{a}^*\hat{c} - i\Delta\hat{a} + i\varepsilon^2\alpha_1\hat{a}\hat{c}\hat{c}^* + i\varepsilon^2\alpha_2\hat{a}^2\hat{a}^* \tag{47a}$$
$$\dot{\hat{c}} = \varepsilon\beta_o\hat{a}^2 - \delta\hat{c} + i\varepsilon^2\beta_1\hat{a}\hat{a}^*\hat{c} \tag{47b}$$

where

$$\alpha_o = (\mu l^3/I)(k_P/2+\omega_o), \quad \alpha_1 = -(\mu l^3/I)^2(k_P/2+\omega_o)^2/(2k_P)$$
$$\alpha_2 = (\mu l^3/I)\hat{\omega}^2(3k_P/2-\omega_o)/k_P{}^2$$
$$\beta_o = 2\hat{\omega}^2/k_P, \quad \beta_1 = 2(\mu l^3/I)\hat{\omega}^2(3k_P/2-\omega_o)/k_P{}^2$$

Let

$$\left.\begin{aligned}
\hat{a} &= r\,e^{i(\Theta-\Delta t)}\\
\hat{c} &= q\,e^{i\Gamma}
\end{aligned}\right\} \tag{48}$$

then Eqs. (47) become

$$\left.\begin{aligned}
\dot{r} &= -\varepsilon\alpha_o rq\cos\lambda\\
\dot{q} &= \varepsilon\beta_o r^2\cos\lambda - \delta q\\
\dot{\Theta} &= \varepsilon\alpha_o q\sin\lambda + \varepsilon^2(\alpha_1 q^2 + \alpha_2 r^2)\\
\dot{\Gamma} &= \varepsilon\beta_o r^2\sin\lambda/q + \varepsilon^2\beta_1 r^2
\end{aligned}\right\} \tag{49}$$

where

$$\lambda = 2\Theta - 2\Delta t - \Gamma$$

and the initial conditions on Eqs. (49) are

$$r = r_{oi}, \quad q = 0, \quad \Theta = 0, \quad \Gamma = 0, \quad \text{at} \quad t = 0 \tag{50}$$

First, let's consider the case of $\delta = 0$ (no damping). Equations (49) may be written as

$$\left.\begin{aligned}
\dot{r} &= -\varepsilon\alpha_o rq\cos\lambda\\
\dot{q} &= \varepsilon\beta_o r^2\cos\lambda\\
\dot{\Theta} &= \varepsilon\alpha_o q\sin\lambda + \varepsilon^2(\alpha_1 q^2 + \alpha_2 r^2)\\
\dot{\Gamma} &= \varepsilon\beta_o r^2\sin\lambda/q + \varepsilon^2\beta_1 r^2
\end{aligned}\right\} \tag{51}$$

Then the solution of Eqs. (51) is obtained in the following form:

$$t = \int \tfrac{1}{2}dQ/[\![Q(\varepsilon^2\beta_o r_o{}^2 - \alpha_o Q)^2 - \{[(2\alpha_2-\beta_1)r_o{}^2\varepsilon^2 - 2\Delta](Q/2) + [2\alpha_1 - (\alpha_o/\beta_o)(2\alpha_2-\beta_1)](Q/2)^2\}^2]\!]^{1/2} \tag{52}$$

where $Q = (\varepsilon q)^2$ and the constant of integration $r_o{}^2$ is proportional to the total energy of the system. Zeros of the quartic expression under the radical in the integrand of Eq. (52) are given approximately by

$$\begin{aligned}
&Q_o = 0, \quad Q_1 = A^2(1 - W/A), \quad Q_2 = A^2(1 + W/A)\\
&Q_3 = \{4\alpha_o/[2(\mu l^3/I)^2(k_P/2+\omega_o)^2/(2k_P)]\}^2
\end{aligned} \tag{53}$$

where

$$A^2 = \varepsilon^2 r_o{}^2(\beta_o/\alpha_o), \qquad (A > 0)$$
$$W^2 = [(A^2/2\alpha_o)(\alpha_1 - 2\Delta/A^2)]^2 \qquad (W > 0)$$

Since the values of Q_3/Q_1 and Q_3/Q_2 are of the order $O[(\varepsilon\hat{\omega}/k_P)^{-2}]$ and physically significant solutions correspond to the range $Q_o \leqq Q \leqq Q_1$, the quartic expression may be approximated by the following cubic expression:

$$\alpha_o{}^2 Q(Q - Q_1)(Q - Q_2) \tag{54}$$

Then from Eq. (52)

$$t = \int_0^Q dQ/\{2\alpha_o[Q(Q-Q_1)(Q-Q_2)]\}^{1/2} \tag{55}$$

and the solutions for $\delta = 0$ are obtained in the following form:

$$\begin{aligned}
r^2 &= r_o{}^2\{1 - \kappa\,\mathrm{sn}^2[\alpha_o At/(\kappa)^{1/2}, \kappa]\}\\
q^2 &= r_o{}^2(\beta_o/\alpha_o)\kappa\,\mathrm{sn}^2[\alpha_o At/(\kappa)^{1/2}, \kappa]
\end{aligned} \tag{56}$$

where

$$\kappa = 1 - (W/A)$$

The least period in t is given by

$$J = 2(\kappa)^{1/2}K(\kappa)/(\alpha_o A) \tag{57}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind

$$K(\kappa) = \int_0^{\pi/2} d\rho/(1 - \kappa^2\sin^2\rho)^{1/2}$$

Next let's consider the case of $\delta \neq 0$. Using Eqs. (49) we obtain the equation

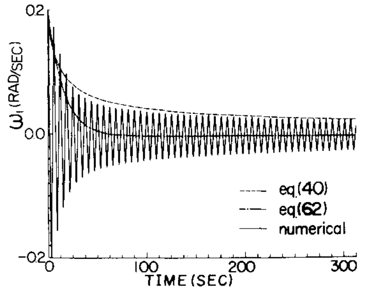

Fig. 4 Plot of ω_1 against time (nonresonance case).

$$(d/dt)[r^2+(\alpha_o/\beta_o)q^2] = -2\delta(\alpha_o/\beta_o)q^2 \tag{58}$$

Substituting Eqs. (56) into Eq. (58) and taking into account the fact that r_o is a function of t, we obtain the following equation:

$$\dot{r}_o = -\delta r_o \kappa \, \mathrm{sn}^2 [\alpha_o A t/(\kappa)^{1/2}, \kappa] \tag{59}$$

Since parameter δ is considered small, it may be used as a small parameter in obtaining a solution of Eq. (59) by the method of averaging. With the averaging operation given by Eq. (23), in the lowest approximation

$$r_o = \hat{r}_o \tag{60}$$

$$\dot{\hat{r}}_o = -(\delta \hat{r}_o/\kappa)[1 - E(\kappa)/K(\kappa)] \tag{61}$$

where $E(\kappa)$ is the complete elliptic integral of the second kind

$$E(\kappa) = \int_0^{\pi/2} (1-\kappa^2 \sin^2 \rho)^{1/2} \, d\rho$$

Neglecting small terms as compared with those of the order $O[(\varepsilon\hat{\omega}/k_P)^2]$, we obtain a solution in the following form:

$$r_o = r_{oi} e^{-\delta^* t} \tag{62}$$

where

$$\delta^* = (\delta/\kappa_{oi})[1 - E(\kappa_{oi})/K(\kappa_{oi})]$$

$$\kappa_{oi} = 1 - W_{oi}/A_{oi}$$

$$A_{oi}^2 = \varepsilon^2 r_{oi}^2 (\beta_o/\alpha_o), \quad W_{oi}^2 = \{[A_{oi}^2/(2\alpha_o)](\alpha_1 - 2\Delta/A_{oi}^2)\}^2$$

It should be noted that the amplitude of nutational body motion decreases exponentially due to the nonlinear internal resonance. This solution is valid so long as the damping of nutational body motion is small: the amplitude must decrease gradually over a time interval of the order of period J, i.e.,

$$J/(1/\delta^*) \ll 1 \tag{63}$$

The condition (63) means that

$$J/(1/\delta^*) \sim [(\kappa)^{1/2} K(\kappa)/(\alpha_o A)]\delta$$
$$\sim (\varepsilon\hat{\omega}/k_P) \ln (\varepsilon\hat{\omega}/k_P) \ll 1$$

since

$$\lim_{\kappa \to 1-0} K(\kappa) = O[\ln (1-\kappa)]$$

Therefore, in the resonance case, the condition (63) is always satisfied.

Figure 4 shows ω_1 vs time for the nonresonance case. It is obvious that Eq. (40) is a good approximate solution for a long-period change of the amplitude of ω_1.

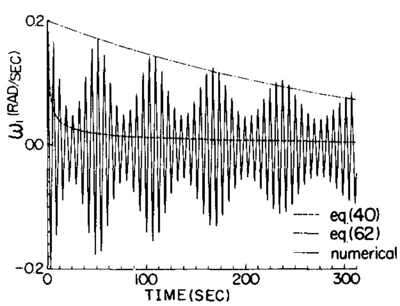

Fig. 5a Plot of ω_1 against time (resonance case).

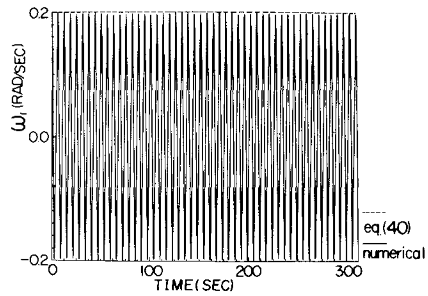

Fig. 5b Plot of ω_1 against time (nonresonance case).

Figure 5a shows a resonance case. In this case it is found that Eq. (62) is a good approximate solution for a long-period change of the amplitude of ω_1. Figure 5b shows the nonresonance motion for large Δ. All other parameters are the same as in Fig. 5a.

It should be noted that the amplitude of ω_1 decays very rapidly due to the resonance between the vibration of the appendages and nutational body motion.

5. Conclusion

The basic characteristics of the dynamics of spin-stabilized satellites with flexible appendages have been investigated from an analytical viewpoint. The method of analysis is based on the method of averaging. It is shown that results obtained by the energy sink method are rigorously derived as the first-order solutions of this method [Eqs. (27) and (40)].

Furthermore, it is shown that this method is applicable to a wide class of spinning satellite having flexible appendages of immediate dynamic significance, to which the energy sink method is not appropriate in application (Figs. 2 and 5a). Closed-form stability criteria (32) are also derived by this method, which show qualitative agreement with those derived by means of more rigorous methods.[5-9]

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